On rough convergence variables of triple sequences

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Nagarajan Subramanian
Alagappa University

Ayhan Esi
Turgut Özal University

Nagarajan Subramanian and Ayhan Esi*

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Abstract: Triple sequence convergence plays an extremely important role in the fundamental theory of mathematics. This paper contains four types of convergence concepts, namely, convergence almost surely, convergence incredibility, trust convergence in mean, and convergence in distribution, and discuss the relationship among them and some mathematical properties of those new convergence.

Keywords: Triple sequences, rough convergence almost surely, convergence in credibility, trust convergence, convergence distribution

MSC 2010: 40A05, 40C99, 40G05

1 Introduction

The idea of rough convergence was first introduced by Phu [14–16] in finite dimensional normed spaces. He showed that the set $\mathrm{LIM}^r x$ is bounded, closed and convex, and introduced the notion of rough Cauchy sequence. He also investigated the relations between rough convergence and other convergence types, and the dependence of $\mathrm{LIM}^r x$ on the roughness of degree r.

Aytar [1] studied rough statistical convergence and defined the set of rough statistical limit points of a sequence. He obtained two statistical convergence criteria associated with this set and proved that this set is closed and convex. Also, Aytar [2] studied the r-limit set of the sequence and proved that it is equal to the intersection of these sets, and that the r-core of the sequence is equal to the union of these sets. Dündar and Çakan [9] investigated the rough ideal convergence and defined the set of rough ideal limit points of a sequence.

In this paper, we introduce the notion of rough convergence and the set of rough limit points of a triple sequence, and we obtain two rough convergence criteria associated with this set. We also prove that this set is closed and convex, and we examine the relations between the set of accumulation points and the set of rough limit points of a triple sequence.

A triple sequence (real or complex) can be defined as a function $x\colon \mathbb{N}\times\mathbb{N}\times\mathbb{N} \to \mathbb{R}(\mathbb{C})$, where $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ denote the set of natural numbers, real numbers and complex numbers, respectively. The different types of notions of triple sequence were introduced and investigated initially by the authors, Dutta, Debnath, Das, Sahiner and many others, see [3–8, 10–13, 17–19].

Let (x_{mnk}) be a triple sequence of rough variables. In this paper we discuss some convergence concepts of rough triple sequences: convergence almost surely, convergence in credibility, trust convergence in mean, and convergence in distribution, as well as the relation between them.

***Corresponding author: Ayhan Esi,** Engineering Faculty, Malatya Turgut Ozal University, 44040, Malatya, Turkey, e-mail: ayhan.esi@ozal.edu.tr
Nagarajan Subramanian, Department of Mathematics, SASTRA University, Thanjavur-613 401, India, e-mail: nsmaths@gmail.com

2 Definitions and preliminaries

Definition 2.1. A triple sequence $x = (x_{mnk})$ is said to be rough convergent (r-convergent) to l, denoted as $x_{mnk} \xrightarrow{r} l$, provided that

$$\forall \epsilon > 0,\ \exists i_\epsilon \in \mathbb{N} : m, n, k \geq i_\epsilon \implies |x_{mnk} - l| < r + \epsilon$$

or, equivalently, if

$$\limsup |x_{mnk} - l| \leq r.$$

Here r is called the roughness of degree. If we take $r = 0$, then we obtain the ordinary convergence of a triple sequence.

Definition 2.2. The triple sequence of rough variables (x_{mnk}) is said to be convergent almost surely to the rough variable l if and only if there exists a set A, with $\mathrm{Tr}(A) = 1$, such that

$$\lim_{u,v,w\to\infty} |x_{mnk}(\lambda) - l(\lambda)| = 0 \quad \text{for every } \lambda \in A.$$

In that case we write $x_{mnk} \xrightarrow{\text{a.s.}} l$.

Definition 2.3. Let r be a non-negative real number and let (x_{mnk}) be a triple sequence of rough variables. We say that the triple sequence (x_{mnk}) converges in trust to the rough variable l if

$$\lim_{m,n,k\to\infty} \mathrm{Tr}\{|x_{mnk} - l| \geq r + \epsilon\} = 0 \quad \text{for every } \epsilon > 0.$$

In that case we write $x_{mnk} \xrightarrow{r\text{-Tr}} l$.

Definition 2.4. Let r be a non-negative real number and let (x_{mnk}) be a triple sequence of rough variables with finite expected values. We say that the triple sequence (x_{mnk}) converges in mean to the rough variable l if

$$\lim_{m,n,k\to\infty} E[|x_{mnk} - l|] = 0.$$

In that case we write $x_{mnk} \xrightarrow{E} l$.

Definition 2.5. Suppose that $\Phi, \Phi_1, \Phi_2, \ldots,$ are the trust distributions of the rough variables $l, l_1, l_2, \ldots,$ respectively. If the triple sequence (Φ_{mnk}) converges weakly to Φ, then we say that $x_{mnk} \xrightarrow{\text{Dis}} l$.

3 Main results

Theorem 3.1. *Let r be a non-negative real number and let (x_{mnk}) be a triple sequence of rough variables. Then $(x_{mnk}) \xrightarrow{a.s.} l$ if and only if for every $\epsilon > 0$, we have*

$$\lim_{u,v,w\to\infty} \mathrm{Tr}\Big\{ \bigcup_{m=u,n=v,k=w} \{|x_{mnk} - l| \geq r + \epsilon\} \Big\} = 0.$$

Proof. For every $m, n, k \geq 1$ and $\epsilon > 0$, we define

$$X = \Big\{\lambda \in \Lambda : \lim_{m,n,k\to\infty} x_{mnk}(\lambda) \neq l(\lambda)\Big\}, \quad X_{mnk} = \{\lambda \in \Lambda : |x_{mnk}(\lambda) - l(\lambda)| \geq r + \epsilon\}.$$

It is clear that

$$X = \bigcup_{\epsilon>0} \Big(\bigcap_{u,v,w=1}^{\infty} \bigcup_{m=u,n=v,k=w}^{\infty} X_{mnk}(r + \epsilon) \Big).$$

Note that $x_{mnk} \xrightarrow{r\text{-a.s.}} l$ if and only if $\mathrm{Tr}(X) = 0$, i.e.,

$$x_{mnk} \xrightarrow{r\text{-a.s.}} l \iff \mathrm{Tr}\Big\{ \bigcap_{u,v,w=1}^{\infty} \bigcup_{m=u,n=v,k=w}^{\infty} X_{mnk}(r + \epsilon) \Big\} = 0 \quad \text{for every } \epsilon > 0.$$

Since

$$\bigcup_{m=u,n=v,k=w}^{\infty} X_{mnk}(r+\epsilon) \downarrow \bigcap_{u,v,w=1}^{\infty} \bigcup_{m=u,n=v,k=w}^{\infty} X_{mnk}(r+\epsilon),$$

it follows from the continuity of the trust measure that

$$\lim_{u,v,w\to\infty} \mathrm{Tr}\Big\{\bigcup_{m=u,n=v,k=w}^{\infty} X_{mnk}(r+\epsilon)\Big\} = \mathrm{Tr}\Big\{\bigcap_{u,v,w=1}^{\infty} \bigcup_{m=u,n=v,k=w}^{\infty} X_{mnk}(r+\epsilon)\Big\}.$$ □

Theorem 3.2. *Let r be a non-negative real number and let (x_{mnk}) be a triple sequence of rough variables. If $(x_{mnk}) \xrightarrow{r\text{-a.s.}} l$, then $(x_{mnk}) \xrightarrow{r\text{-Tr}} l$.*

Proof. It follows from the a.s. convergence and Theorem 3.1 that

$$\lim_{u,v,w\to\infty} \mathrm{Tr}\Big\{\bigcup_{m=u,n=v,k=w}^{\infty} \{|x_{mnk} - l| \geq r+\epsilon\}\Big\} = 0$$

for each $\epsilon > 0$. Since, for every $u, v, w \geq 1$, we have

$$\{|x_{uvw} - l| \geq r+\epsilon\} \subset \bigcup_{m=u,n=v,k=w}^{\infty} \{|x_{mnk} - l| \geq r+\epsilon\},$$

the theorem holds. □

Theorem 3.3. *Let r be a non-negative real number and let (x_{mnk}) be a triple sequence of rough variables. If $(x_{mnk}) \xrightarrow{r\text{-mean}} l$, then $(x_{mnk}) \xrightarrow{r\text{-Tr}} l$.*

Proof. For any given number $\epsilon > 0$,

$$\mathrm{Tr}\{|x_{mnk} - l| \geq r+\epsilon\} \leq \frac{E[|x_{mnk} - l|]}{r+\epsilon} \to 0 \quad \text{as } m,n,k \to \infty.$$

Thus, $(x_{mnk}) \xrightarrow{r\text{-Tr}} l$.

Hence, the trust distribution $\Phi\colon \mathbb{R}^3 \to [0,1]$ of a rough variable l is defined by

$$\Phi(x) = \mathrm{Tr}\{\lambda \in \Lambda : l(\lambda) \leq x\} \quad \text{for all } x \in \mathbb{R}^3,$$

i.e., $\Phi(x)$ is the trust of the rough variable $l \leq x$. □

Theorem 3.4. *Suppose that $l, l_1, l_2, \ldots,$ are rough variables. If $(x_{mnk}) \xrightarrow{r\text{-Tr}} l$, then $(x_{mnk}) \xrightarrow{r\text{-Dist}} l$.*

Proof. Let x be any given continuity point of the distribution Φ. On the one hand, for any $y > x$, we have

$$\{x_{mnk} \leq x\} = \{x_{mnk} \leq x, l < y\} \bigcup \{x_{mnk} \leq x, l > y\} \subset \{l < y\} \bigcup \{|x_{mnk} - l| \geq y - x\},$$

which implies

$$\Phi_{mnk}(x) \leq \Phi(y) + \mathrm{Tr}\{|x_{mnk} - l| \geq y - x\}.$$

Since $(x_{mnk}) \xrightarrow{r\text{-Tr}} l$, we have

$$\mathrm{Tr}\{|x_{mnk} - l| \geq y - x\} \to 0.$$

Thus, we obtain

$$\lim_{m,n,k\to\infty} \sup \Phi_{mnk}(x) \leq \Phi(y) \quad \text{for every } y > x.$$

Letting $y \to x$, we get

$$\lim_{m,n,k\to\infty} \sup \Phi_{mnk}(x) \leq \Phi(x). \tag{3.1}$$

On the other hand, for any $z < x$, we have

$$\{l \leq z\} = \{l \leq z, x_{mnk} \leq x\} \bigcup \{l \leq z, x_{mnk} > x\} \subset \{x_{mnk} \leq x\} \bigcup \{|x_{mnk} - l| \geq x - z\},$$

which implies

$$\Phi(z) \le \Phi_{mnk}(x) + \mathrm{Tr}\{|x_{mnk} - l| \ge x - z\}.$$

Since $\mathrm{Tr}\{|x_{mnk} - l| \ge x - z\} \to 0$, we obtain

$$\Phi(z) \le \lim_{mnk\to\infty} \inf \Phi_{mnk}(x) \quad \text{for any } z < x.$$

Letting $z \to x$, we get

$$\Phi(x) \le \lim_{m,n,k\to\infty} \inf \Phi_{mnk}(x). \tag{3.2}$$

It follows from (3.1) and (3.2), we get $\Phi_{mnk}(x) \to \Phi(x)$. □

Conclusion. This paper contributed to the research area of rough variable triple sequences, which were defined and then some mathematical properties of them were discussed.

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