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Kübik Hermityen ve Kübik Bezier Eğrileri Tarafından Oluşturulan Dönel Yüzeyler

Araştırma Makalesi / Research Article

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ÖZ

Dönel yüzeylerin şekillerinin ayarlanmasında geometrik tasarımın istenilen şekilde olması için, ilk olarak iki yerel şekil parametreli kübik Hermityan ve kübik Bezier eğrileri kullanılarak dönel yüzeyler oluşturuldu. Oluşturulan bu yeni dönel yüzeylerin, yerel şekil parametrelerinin değiştirilmesi ile yüzeylerin şekillerinin ayarlaması konusunda iyi bir performansa sahip olduğu görüldü. Ayrıca, kübik Hermityan ve kübik Bezier eğrileri tarafından oluşturulan dönel yüzeyler, ilgi çekici yüzeylerin tasarımı için değerli bir yol sağlamaktadır. Bu bağlamda, bu dönel yüzeylerin ortalama ve Gauss eğrilikleri elde edilerek, bu yüzeyler için bazı karakterizasyonlar verildi.

Anahtar Kelimeler: Hermityen eğrileri, bezier eğrileri, dönel yüzeyler, şekil parametresi.

Rotational Surfaces Generated by Cubic Hermitian and Cubic Bezier Curves

ABSTRACT

To tackle the geometric design in adjusting shapes of rotation surfaces, firstly the rotation surfaces have been constructed by using cubic Hermitian and cubic Bezier curves with two local shape parameters. It has been seen that, the new rotational surfaces which have been constructed have a good performance on adjusting their shapes by changing the local shape parameters. Also, the rotational surfaces generated by cubic Hermitian and cubic Bezier curves have provided a valuable way for the design of interesting surfaces. In this context, some characterizations have been given for these rotational surfaces obtaining the mean and Gaussian curvatures of them.

Keywords: Hermitian curves, bezier curves, rotational surfaces, shape parameter.

1. INTRODUCTION

Geometry of curves plays an important role in industrial design and engineering as well as being an important branch of mathematics. In recent years, many authors such as G. Farin, J. Hoschek and A. Saxena have worked on the structure of curves for mathematical modelling [5,6,10]. The most important of these curves are the Hermitian curves, Ferguson curves, Bezier curves and etc. The De Casteljau algorithm has shown that, Bezier curves are written as linear combinations of Bernstein polynomials (for detail about these curves, see [6,9,10]).

Also, the geometry of surfaces such as, rotational surfaces, ruled surfaces, rational Bezier surfaces, rational B-spline surfaces, non-uniform rational B-spline surfaces, discrete surfaces and etc. have been studied by geometers and engineers widely in Euclidean space, Minkowski space, Galilean space, pseudo-Galilean space and etc [1,2,3,4,5,7,8].

For example, E. Octafiatiningsih and I. Sujarwo have used Quadratic Bezier curve on rotational and symmetrical lampshade in [9]. So, by using this work we have constructed the present paper which is divided into three steps as follows:

- **i.** Recalling cubic Hermitian and cubic Bezier curves;
- **ii.** rotating cubic Hermitian curve and cubic Bezier curve about an axis to produce geometric surface designs;
- **iii.** giving some characterizations for these rotational surfaces obtaining the mean and Gaussian curvatures of them.

Consequently, the aim of this study is modelling some industrial objects by constructing and rotating cubic Hermitian and cubic Bezier curves and also giving new ideas for producers about object modelling industry.

2. PRELIMINARIES

A cubic Hermitian curve is a cubic polinomial curve segment constrained to a given position p and a tangent vector v at each endpoints.

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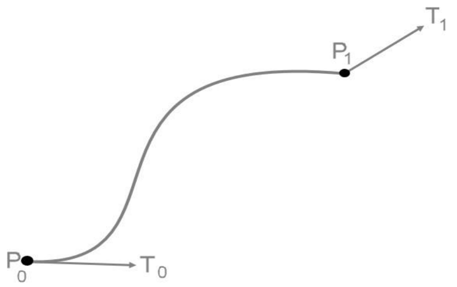


Figure 1. Cubic Hermitian curve created with two control points and two tangent segments

First, we'll recall the parametric expression of a cubic Hermitian curve [6,10].

A parametric cubic curve $P(u)$ in Euclidean 3-space is defined as $\mathrm{P(u) = (x(u), y(u), z(u))}$, where

$$\begin{aligned} x(u) &= a_x + b_x u + c_x u^2 + d_x u^3, \\ y(u) &= a_y + b_y u + c_y u^2 + d_y u^3, \\ z(u) &= a_z + b_z u + c_z u^2 + d_z u^3, \end{aligned} \quad (1)$$

with parameters bounded in intervals $0 \leq u \leq 1$. Then, we can write it as

$$P(u) = \big(x(u), y(u), z(u)\big) = a + bu + cu^2 + du^3. \quad (2)$$

Then, for $u = 0$ and $u = 1$, we have

$$\begin{aligned} P(u = 0) &= a, \\ P(u = 1) &= a + b + c + d, \\ P'(u = 0) &= b, \\ P'(u = 1) &= b + 2c + 3d \end{aligned} \quad (3)$$

with a_x, b_x, c_x and d_x are algebraic scalar coefficients.

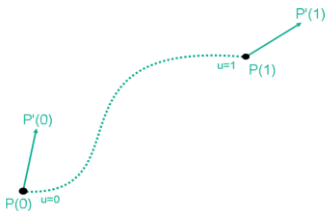


Figure 2. Control points and tangent segments of cubic Hermitian curve for u = 0 and u = 1

If the system (3) is solved, then the values of the vectors a, b, c and d are obtained by

$$\begin{aligned} a &= P(0), \\ b &= P'(0), \\ c &= -3P(0) + 3P(1) - 2P'(0) - P'(1), \\ d &= 2P(0) - 2P(1) + P'(0) + P'(1). \end{aligned} \quad (4)$$

If we use the equations (4) in (2), then the Hermitian curve is obtained as:

$$P(u) = P(0)H_1(u) + P(1)H_2(u) + P'(0)H_3(u) + P'(1)H_4(u), \quad (5)$$

where $H_1(u), H_2(u), H_3(u)$ and $H_4(u)$ are the base functions (or blending functions) of Hermitian curve given by

$$\begin{aligned} H_1(u) &= 1 - 3u^2 + 2u^3, \\ H_2(u) &= 3u^2 - 2u^3, \\ H_3(u) &= u - 2u^2 + u^3, \\ H_4(u) &= -u^2 + u^3 \end{aligned} \quad (6)$$

and $P(0)$, $P(1)$, $P'(0)$ and $P'(1)$ are geometric coefficients.

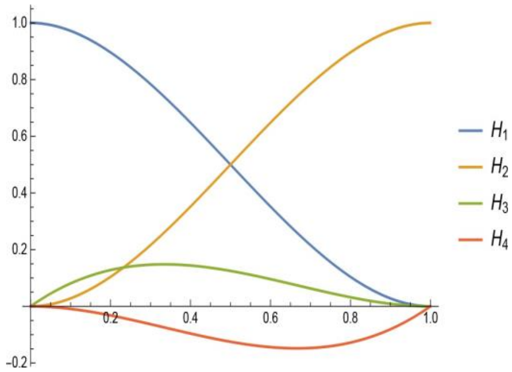


Figure 3. Hermitian blending functions

For the blending functions of Hermitian curve we have the following:

At $u = 0$ and $u = 1$, we get

$$H_1 = 1, H_2 = H_3 = H_4 = 0; P(0) = P_0,$$
$$H_1' = H_2' = H_4' = 0, H_3' = 1; P'(0) = T_0,$$

and

$$H_1 = H_3 = H_4 = 0, H_2 = 1; P(1) = P_1,$$
$$H_1' = H_2' = H_3' = 0, H_4' = 1; P'(1) = T_1,$$

respectively. This gives us the endpoints and tangent vectors at endpoints by using blending functions.

Also, by putting the blending functions we can give the matrix form of cubic Hermitian curves as follows:

$$H = [H_1(u), H_2(u), H_3(u), H_4(u)]$$

$$= [u^3 \;\; u^2 \;\; u \;\; 1] \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} = UM_H, \quad (7)$$

where M_H is called the Hermitian characteristic matrix.

Collecting the Hermitian geometric coefficients into a geometric vector B, we have a matrix formulation for the Hermitian curve $P(u)$ as

$$P(u) = UM_H B, \quad (8)$$

where

$$B = \begin{bmatrix} P(0) \\ P(1) \\ P'(0) \\ P'(1) \end{bmatrix}.$$

M_H transforms geometric coordinates from the Hermitian bases to the algebraic coefficients of the monomial bases.

Next, let us recall some notations about the cubic Bezier curve [6,10].

n-th degree of Bezier curve is defined as (n+1) control points' weighted linear combination using Bernstein polynomials. A Bezier curve can be expressed by

$$P(u) = \sum_{i=0}^{n} C_i^n (1-u)^{n-i} u^i P_i = \sum_{i=0}^{n} B_i^n(u) P_i, \; 0 \le u \le 1,$$

where $B_i^n(u)$ is called Bernstein polynomials. More specifically, we can examine the behavior of Bezier curve for $3rd$ degree polynomials as follows:

Let $P(u)$ be the cubic Bezier curve lying on xz-plane. It has 4 control points P_i $(i = 0, 1, 2, 3)$ and four base functions $f_i(u)$ $(i = 0, 1, 2, 3)$ with parameters bounded in intervals $0 \le u \le 1$. Then we can write it as

$$P(u) = \sum_{i=0}^{3} f_i(u) P_i = f_0(u)P_0 + f_1(u)P_1 + f_2(u)P_2 + f_3(u)P_3$$

with the base functions

$$\begin{aligned} f_0(u) &= 1 - 3u + 3u^2 - u^3, \\ f_1(u) &= 3u - 6u^2 + 3u^3, \\ f_2(u) &= 3u^2 - 3u^3, \\ f_3(u) &= u^3. \end{aligned} \tag{9}$$

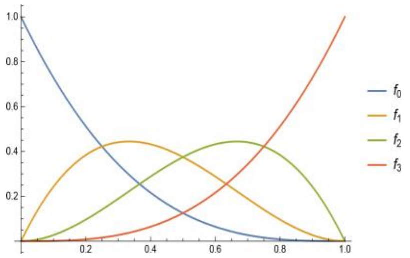


Figure 4. Cubic Bezier Base Functions

Also, according to the base functions, we can give the matrix form of cubic Bezier curve as follows:

$$B = [f_1(u), f_2(u), f_3(u), f_4(u)]$$

$$= [u^3 \;\; u^2 \;\; u \;\; 1] \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} = UM_B, \tag{10}$$

where M_B is called the Bezier matrix.

Collecting the Bezier geometric coefficients into a geometric vector G which is defined by the user, is an array of data points. Here, we have a matrix formulation for the Bezier curve $P(u)$ as

$$P(u) = UM_B G, \tag{11}$$

where

$$G = \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{bmatrix}.$$

Note that the curve does not pass through the points P_1 and P_2. In cubic Bezier segments, in order to change the curve's shape we may relocate any of control points P_0, P_1, P_2 or P_3. We also know that, for Hermitian segments, we have to specify end slopes for a particular shape and this situation is difficult for researchers. Furthermore, Bezier curve is easier to specify the shape of control polyline than Hermitian curve.

For more details about Hermitian and Bezier curves, we refer to [6,9,10].

Now, let us investigate the rotational surfaces according to the axes of rotation in E^3.

Rotation is the change of an object coordinates into the new position by moving the whole coordinate points defined in the initial form with an angle about an axis of rotation. The coordinate system E^3 has three rotation axes. First suppose that the axis of rotation is the z-axis.

Let A be a 3×3 regular matrix and $0 \neq \xi \in E^3$ be a vector. If A satisfies the following conditions, then it is said that A denotes a rotation in positive direction

i. $A\xi = \xi$,

ii. $AIA^t = I$,

iii. $\det A = 1$,

where I is the 3×3 unit matrix.

From this definition, it can be seen that the rotation matrix which fixes the z-axis is the set of 3×3 matrices defined by

$$A(v) = \begin{bmatrix} \cos v & -\sin v & 0 \\ \sin v & \cos v & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad v \in \mathbb{R}$$

Then, by rotating the curve

$\alpha(u) = (\alpha_1(u), \alpha_2(u), \alpha_3(u))$ about the z-axis, the rotational surface M can be parametrized by

$$\begin{aligned} \Psi(u, v) = (\alpha_1(u)\cos v - \alpha_2(u)\sin v, \alpha_1(u)\sin v \\ + \alpha_2(u)\cos v, \alpha_3(u)). \end{aligned} \tag{12}$$

By rotating the curve α about the x-axis and y-axis, one can write the rotational surfaces similarly.

3. CONSTRUCTION OF ROTATIONAL SURFACES GENERATED BY CUBIC HERMITIAN CURVE

In this section, we'll construct the rotational surface generated by cubic Hermitian curve by using the structure of a tube deformation.

Suppose given a tube of radius r, where $r \in [a, b]$, i.e. the minimum radius of the tube is a and the maximum of the radius is b. Also, let we define the height of the tube as h, where $h \in [c, d]$, i.e. the minimum height of the tube is c while the maximum of the radius is d. The selection of the value of r and h in the interval aims to differences in size of shape components of geometric design.

Firstly, we determine a center point on the tube base circle $(x_1, y_1, z_1) = (0,0,0)$. Then, for this center point and $v = 0$ the tube base circle using the circle equation is built and the point $P(0)$ is given by

$$(x_1 + r_1 \cos v, y_1 + r_1 \sin v, z_1) = (r_1, 0, 0). \tag{13}$$

Also, let the center point on the tube roof circle be $(x_1, y_1, z_1) = (0,0,h)$. Then, for $v = 0$, we can build tube roof circle using the circle equation and obtain the point, namely $P(1)$ as

$$(x_1 + r_2 \cos v, y_1 + r_2 \sin v, z_1) = (r_2, 0, h). \quad (14)$$

Then, for controlling the curvatures of the Hermitian curve, we can determine the control points $P'(0)$ and $P'(1)$ as follows:

$$P'(0) = (x, 0, 0) \quad (15)$$

and

$$P'(1) = (x, 0, z), \quad (16)$$

where $-2r \leq x, z \leq 2h$ and $x, z \in \mathbb{R}$.

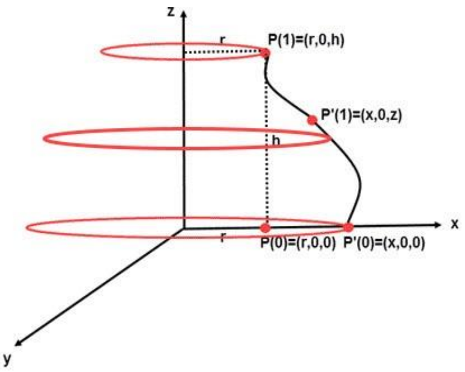


Figure 5. Representation of a tube deformation for Hermitian curve

Further by using (13)-(16) in the equation $P(u) = P(0)H_1(u) + P(1)H_2(u) + P'(0)H_3(u) + P'(1)H_4(u)$, the Hermitian curve is obtained by

$$P(u) = \big(r_1 H_1(u) + r_2 H_2(u) + x H_3(u) + x H_4(u), 0, h H_2(u) + z H_4(u)\big), \ 0 \leq u \leq 1 \quad (17)$$

with the blending functions (6).

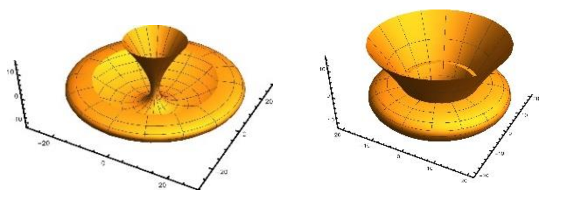
3.1. Some Characterizations of Rotational Surfaces Generated By Cubic Hermitian Curve

In this subsection, firstly we'll give some examples for rotational surface generated by cubic Hermitian curve by obtaining the parametric expression of it. Also, we'll give some characterizations for it with the aid of the mean and Gaussian curvatures.

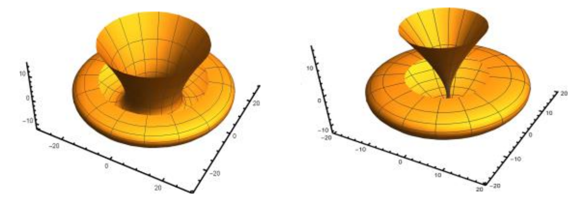
By rotating the Hermitian curve (17) around z-axis, we get the rotational surface as

$$\begin{aligned} P(u,v) = (&[r_1(1-3u^2+2u^3) + r_2(3u^2-2u^3) \\ &+ x(u-3u^2+2u^3)] \cos v, \\ &[r_1(1-3u^2+2u^3) + r_2(3u^2-2u^3) \\ &+ x(u-3u^2+2u^3)] \sin v, \\ &h(3u^2-2u^3) + z(-u^2+u^3)). \end{aligned} \quad (18)$$

In the following figures, one can see the rotational surface (18) for $x = 100, \ h = 15, \ z = 150$ and different radius r_1 and r_2:



(A) $r_1 = 20, r_2 = 10$ (B) $r_1 = 10, r_2 = 20$



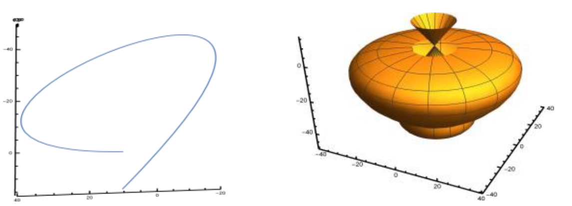
(C) $r_1 = r_2 = 20$ (D) $r_1 = r_2 = 10$

Figure 6. Rotational surface (18) for different radius

Now, by taking $r_1 = r_2 = r$ in (18), we can write the rotational surface generated by cubic Hermitian curve as

$$\begin{aligned} P(u,v) = (&[r + x(u-3u^2+2u^3)] \cos v, \\ &[r + x(u-3u^2+2u^3)] \sin v, \\ &h(3u^2-2u^3) + z(-u^2+u^3)). \end{aligned} \quad (19)$$

In the following figures, one can see the Hermitian curve and rotational surface (19) generated by this cubic Hermitian curve for $r = 10, \ x = 300, h = 15, z = 400$:



(A) Hermitian curve (B) Rotational surface

Figure 7. Hermitian curve and rotational surface (19)

The coefficients of the first and second fundamental forms of the rotational surface (19) are obtained as

$$\begin{aligned} E &= x^2(1-6u+6u^2)^2 \\ &\quad + [6hu(1-u) + zu(-2+3u)]^2, \\ F &= 0, \\ G &= [r + x(u-3u^2+2u^3)]^2 \end{aligned} \quad (20)$$

and

$$\begin{aligned} L &= \frac{2x}{\sqrt{D}}[3h(1-2u) - z(1-3u+3u^2)], \\ M &= 0, \\ N &= \frac{1}{\sqrt{D}}[r + x(u-3u^2+2u^3)] \times \\ &\quad [6hu(1-u) + zu(-2+3u)], \end{aligned} \quad (21)$$

respectively. Here, the unit normal of the surface is

$$N(u,v) = \frac{1}{\sqrt{D}}\big(-[6hu(1-u)+zu(-2+3u)]\cos v,\ -[6hu(1-u)+zu(-2+3u)]\sin v,\ x(1-6u+6u^2)\big)$$

and

$$D = [6hu(1-u)+zu(-2+3u)]^2 + x^2(1-6u+6u^2)^2.$$

So, we can give the following Theorem:

Theorem 1. *The mean curvature and Gaussian curvature of the rotational surface (19) are*

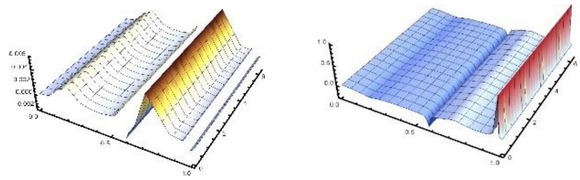
$$H = \frac{\begin{array}{c} 2x[r+x(u-3u^2+2u^3)] \times \\ [3h(1-2u)-z(1-3u+3u^2)] + \\ [x^2(1-6u+6u^2)^2+[6hu(1-u)+zu(-2+3u)]^2] \times \\ [6hu(1-u)+zu(-2+3u)] \end{array}}{\begin{array}{c} 2[x^2(1-6u+6u^2)^2+[6hu(1-u)+zu(-2+3u)]^2]^{\frac{3}{2}} \times \\ [r+x(u-3u^2+2u^3)] \end{array}}$$

and *(22)*

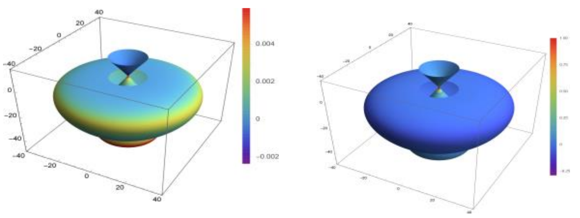
$$K = \frac{\begin{array}{c} 2x[3h(1-2u)-z(1-3u+3u^2)] \times \\ [6hu(1-u)+zu(-2+3u)] \end{array}}{\begin{array}{c} [x^2(1-6u+6u^2)^2+[6hu(1-u)+zu(-2+3u)]^2]^2 \times \\ [r+x(u-3u^2+2u^3)] \end{array}}, \quad (23)$$

respectively.

The following figures show the Gaussian and mean curvatures functions' graphics of the rotational surface (19) for $r = 10,\ x = 300,\ h = 15,\ z = 400$ and the variations of Gaussian and mean curvatures on this surface:



(A) Gaussian curvature function's graphic (B) Mean curvature function's graphic



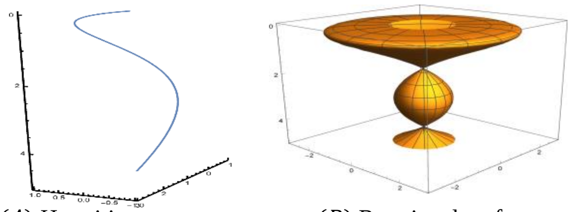
(C) Variation of Gaussian curvature on surface (D) Variation of mean curvature on surface

Figure 8. Gaussian and mean curvatures' graphics and the variations of Gaussian and mean curvatures on surface

Now, let us take $z = 2h$ in the equations (19)-(23). Then, we have

$$P(u,v) = ([r+x(u-3u^2+2u^3)]\cos v,\ [r+x(u-3u^2+2u^3)]\sin v,\ hu^2). \quad (24)$$

In the following figures, one can see the Hermitian curve and rotational surface (24) generated by this cubic Hermitian curve for $r = 1,\ x = 20,\ h = 5,\ z = 10$:



(A) Hermitian curve (B) Rotational surface

Figure 9. Hermitian curve and rotational surface (24)

Also, for (24)

$$E = x^2(1-6u+6u^2)^2 + 4h^2u^2,\quad F = 0,$$
$$G = [r+x(u-3u^2+2u^3)]^2 \quad (25)$$

and

$$L = \frac{2hx}{\sqrt{D}}(1-6u^2),\quad M = 0,$$
$$N = \frac{2hu}{\sqrt{D}}[r+x(u-3u^2+2u^3)]. \quad (26)$$

Theorem 2. *The mean curvature and Gaussian curvature of the rotational surface (24) are*

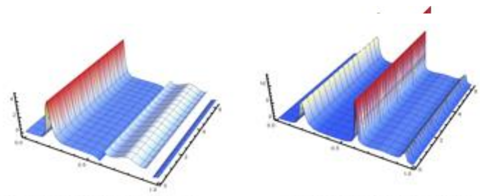
$$H = \frac{h\{x(1-6u^2)[r+x(u-3u^2+2u^3)]\} + hu\left[x^2(1-6u+6u^2)^2+4h^2u^2\right]}{[x^2(1-6u+6u^2)^2+4h^2u^2]^{\frac{3}{2}}[r+x(u-3u^2+2u^3)]}$$

and *(27)*

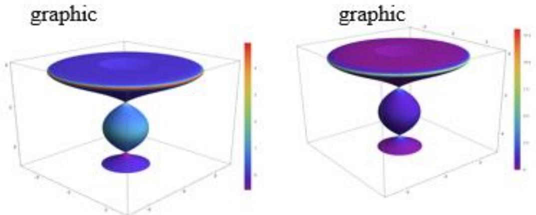
$$K = \frac{4h^2xu(1-6u^2)}{[x^2(1-6u+6u^2)^2+4h^2u^2]^2[r+x(u-3u^2+2u^3)]}, \quad (28)$$

respectively.

The following figures show the Gaussian and mean curvatures functions' graphics of the rotational surface (24) for $r = 1,\ x = 20,\ h = 5,\ z = 10$ and the variations of Gaussian and mean curvatures on this surface:



(A) Gaussian curvature's graphic (B) Mean curvature's graphic



(C) Variation of Gaussian curvature on surface (D) Variation of mean curvature on surface

Figure 10. Gaussian and mean curvatures' graphics and the variations of Gaussian and mean curvatures on surface

Now, from (27) and (28) we can give the following characterizations:

Corollary 1. *Let M be the rotational surface (24) which is generated by cubic Hermitian curve.*

- ***i.*** *Then, the mean curvature of surface cannot vanish at the initial point of the Hermite curve.*
- ***ii.*** *Then, the mean curvature of surface vanishes at the ending point of the Hermite curve if and only if the equation $5hrx = x^2 + 4h^2$ holds.*
- ***iii.*** *If the mean curvature of surface vanishes at the ending point of the Hermite curve, then the control point of $P'(1)$ cannot be on the z-axis or the control point of $P'(0)$ cannot be on the origin.*

Corollary 2. *Let M be the rotational surface (24) which is generated by cubic Hermitian curve. Then, the Gaussian curvature of surface*

- ***i.*** *vanishes at the initial point of the Hermite curve;*
- ***ii.*** *vanishes on the parametric curve $P(\frac{1}{\sqrt{6}}, v)$ of M;*
- **iii.** *vanishes, if the control point of $P'(1)$ is on the z-axis or the control point of $P'(0)$ is on the origin.*

4. CONSTRUCTION OF ROTATIONAL SURFACES GENERATED BY CUBIC BEZIER CURVE

In this section, we'll construct the rotational surface generated by cubic Bezier curve by using the structure of a tube deformation.

Suppose given a tube of radius r_1 and r_2, where $r_1, r_2 \in [a, b]$. Also, let we define the height of the tube as h, where $h \in [c, d]$. The selection of the value of r_1, r_2 and h in the interval aims to differences in size of shape components of geometric design.

Firstly, we determine a center (initial) point on the tube base circle $(x_1, y_1, z_1) = (0,0,0)$. Then, for this center (initial) point and $v = 0$ the tube base circle using the circle equation is built and the point P_0 is given by

$$P_0 = (x_1 + r_1 \cos v, y_1 + r_1 \sin v, z_1) = (r_1, 0,0). \quad (29)$$

Also, let the center (ending) point on the tube roof circle be $(x_1, y_1, z_1) = (0,0,h)$. Then, for $v = 0$, we can build tube roof circle using the circle equation and obtain the point, namely P_3 as

$$P_3 = (x_1 + r_2 \cos v, y_1 + r_2 \sin v, z_1) = (r_2, 0, h). \quad (30)$$

Then, the other two control points P_1 and P_2 of the cubic Bezier curve can be defined as follows:

$$P_1 = (x_1, 0, z_1) \quad (31)$$

and

$$P_2 = (x_2, 0, z_2). \quad (32)$$

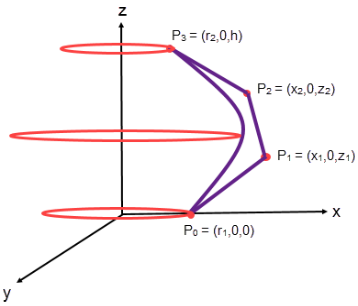


Figure 11. Representation of a tube deformation for Bezier curve

Further by using (29)-(32) in the equation

$$P(u) = f_0(u)P_0 + f_1(u)P_1 + f_2(u)P_2 + f_3(u)P_3,$$

the Bezier curve is obtained by

$$P(u) = (r_1 f_0(u) + x_1 f_1(u) + x_2 f_2(u) + r_2 f_3(u), 0, z_1 f_1(u) + z_2 f_2(u) + h f_3(u)), 0 \le u \le 1 \quad (33)$$

with the base functions (9).

4.1. Some Characterizations of Rotational Surfaces Generated By Cubic Bezier Curve

In this subsection, firstly we'll give some examples for rotational surface generated by cubic Bezier curve by obtaining the parametric expression of it. Also, we'll give some characterizations for it with the aid of the mean and Gaussian curvatures.

By rotating the Bezier curve (33) around z-axis, we get the rotational surface $P(u, v)$ with two local shape parameters as

$$\begin{aligned}([r_1(1 - 3u + 3u^2 - u^3) + x_1(3u - 6u^2 + 3u^3) \\ + x_2(3u^2 - 3u^3) + r_2 u^3] \cos v, \\ [r_1(1 - 3u + 3u^2 - u^3) + x_1(3u - 6u^2 + 3u^3) \\ + x_2(3u^2 - 3u^3) + r_2 u^3] \sin v, \\ z_1(3u - 6u^2 + 3u^3) + z_2(3u^2 - 3u^3) + u^3 h).\end{aligned} \quad (34)$$

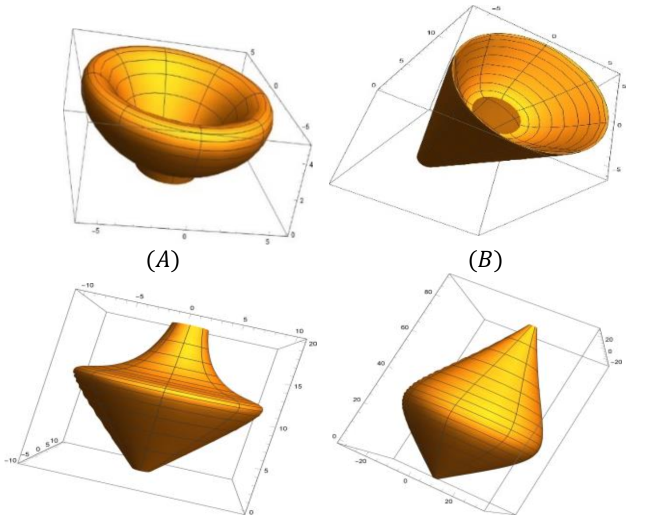
In the Figure 12, one can see the rotational surface (34) for different values of $r_1, r_2, x_1, x_2, z_1, z_2$ and h which have been choosed as following, respectively:

(A) $r_1 = 2,\ r_2 = 0,\ x_1 = 5,\ x_2 = 10,\ z_1 = 10,\ z_2 = 2$ and $h = 0.1$;

(B) $r_1 = 1,\ r_2 = 2,\ x_1 = 10,\ x_2 = 5,\ z_1 = 25,\ z_2 = 10$ and $h = 1$;

(C) $r_1 = 1,\ r_2 = 2,\ x_1 = 20,\ x_2 = 5,\ z_1 = 20,\ z_2 = 5$ and $h = 20$;

(D) $r_1 = 1,\ r_2 = 2,\ x_1 = 50,\ x_2 = 35, z_1 = 40,\ z_2 = 25$ and $h = 90$.



(A) (B)

(C) (D)

Figure 12. Rotational surface (34) for different values of $r_1, r_2, x_1, x_2, z_1, z_2$ and h

The first derivative of the surface (34) with respect to u and v are

$$P_u(u,v) = ([r_1(-3+6u-3u^2)+x_1(3-12u+9u^2) + x_2(6u-9u^2)+3u^2r_2]\cos v, [r_1(-3+6u-3u^2)+x_1(3-12u+9u^2) + x_2(6u-9u^2)+3u^2r_2]\sin v, z_1(3-12u+9u^2)+z_2(6u-9u^2)+3u^2h)$$

$$P_v(u,v) = (-\rho\sin v, \rho\cos v, 0)$$

where $\rho = r_1(1-3u+3u^2-u^3)+x_1(3u-6u^2+3u^3)+x_2(3u^2-3u^3)+r_2u^3,\ 0\le u\le 1.$ So the coefficients of first fundamental form of rotational surface (34) are obtained as

$$E = \beta^2+\theta^2, \quad F = 0, \quad G = \rho^2, \tag{35}$$

where

$$\beta = r_1(-3+6u-3u^2)+x_1(3-12u+9u^2) + x_2(6u-9u^2)+3u^2r_2,$$

$$\theta = z_1(3-12u+9u^2)+z_2(6u-9u^2)+3u^2h.$$

Also, the unit normal of the surface can be found as

$$N(u,v) = \frac{1}{\sqrt{\theta^2+\beta^2}}(-\theta\cos v, -\theta\sin v, \beta).$$

The second derivatives of $P(u,v)$ are given by

$$P_{uu}(u,v) = ([r_1(6-6u)+x_1(-12+18u) + x_2(6-18u)+6ur_2]\cos v, [r_1(6-6u)+x_1(-12+18u) + x_2(6-18u)+6ur_2]\sin v, z_1(-12+18u)+z_2(6-18u)+6uh),$$

$$P_{uv}(u,v) = (-\beta\sin v, \beta\cos v, 0),$$

$$P_{vv}(u,v) = (-\rho\cos v, -\rho\sin v, 0).$$

Then, the coefficients of second fundamental form of rotational surface are obtained by

$$L = \frac{-\sigma\theta+\mu\beta}{\sqrt{\theta^2+\beta^2}}, \quad M = 0, \quad N = \frac{\rho\theta}{\sqrt{\theta^2+\beta^2}}, \tag{36}$$

respectively. Here,

$$\sigma = r_1(6-6u)+x_1(-12+18u)+x_2(6-18u)+6ur_2$$

and

$$\mu = z_1(-12+18u)+z_2(6-18u)+6uh.$$

We have the following theorem.

Theorem 3. *The mean curvature and Gaussian curvature of the rotational surface (34) are*

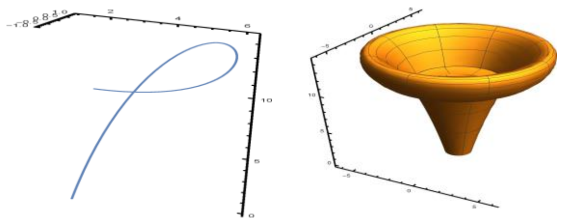
$$H = \frac{\theta(\beta^2+\theta^2)+\rho(\mu\beta-\sigma\theta)}{2\rho(\beta^2+\theta^2)^{\frac{3}{2}}} \tag{37}$$

and

$$K = \frac{\theta(\mu\beta-\sigma\theta)}{\rho(\beta^2+\theta^2)^2}, \tag{38}$$

respectively.

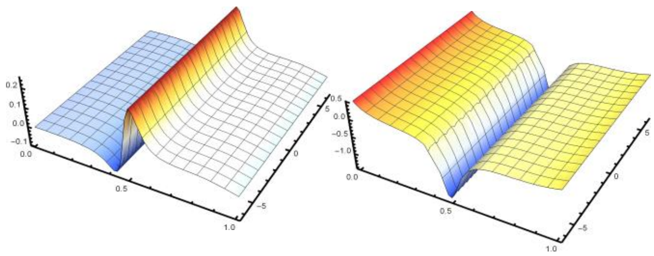
In the following figures, one can see the cubic Bezier curve and rotational surface (34) generated by this cubic Bezier curve for $r_1 = 1,\ r_2 = 2,\ x_1 = 5,\ x_2 = 10,\ z_1 = 25,\ z_2 = 10,\ h = 10$:



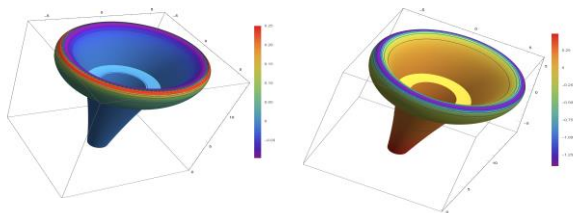
(A) Bezier curve (B) Rotational surface (34)

Figure 13. Bezier curve and rotational surface (34)

The following figures show the Gaussian and mean curvatures functions' graphics of the rotational surface (34) and the variations of Gaussian and mean curvatures on this surface:



(A) Gaussian curvature function's graphic (B) Mean curvature function's graphic



(C) Variation of Gaussian curvature on surface

(D) Variation of mean curvature on surface

Figure 14. Gaussian and mean curvatures' graphics and the variations of Gaussian and mean curvatures on surface

Now, from (37) and (38), let us give some characterizations for rotational surface (34) generated by cubic Bezier curve.

Corollary 3. *Let M be the rotational surface (34) which is generated by cubic Bezier curve.*

i. *If the control point of P_1 is on the x-axis, then the mean curvature of the surface cannot vanish at the initial point of the Bezier curve.*

ii. *If the control point of P_1 is on the origin, then the mean curvature of the surface cannot vanish at the initial point of the Bezier curve.*

iii. *If the control point of P_1 is on the x-axis, then the Gaussian curvature of the surface vanishes at the initial point of the Bezier curve.*

iv. *If the control point of P_2 is on the origin, then the Gaussian curvature of the surface cannot vanish at the initial point of the Bezier curve.*

v. *If the control points of P_2 and P_3 are on the x-axis, then the mean curvature of the surface cannot vanish at the ending point of the Bezier curve.*

vi. *If the control points of P_2 and P_3 are on the x-axis, then the Gaussian curvature of the surface vanishes at the ending point of the Bezier curve.*

vii. *If the control point of P_2 is on the origin and the equation $x_1 h = z_1 r_2$, then the Gaussian curvature of the surface vanishes at the ending point of the Bezier curve.*

5. CONCLUSION

In the present study, we have used Hermitian and Bezier curves in the cubic structure and applied a rotation to these curves. Based on these curves, we have obtained geometric shapes as the result of rotating surfaces. Industrial objects such as lampshades, vases, bullets, etc. provide less costly, more convenient and more reliable results through the use of these structures. In this context, by changing the values of constants r_1, r_2, x_1, x_2, z_1, z_2 and h, different curves and rotational surfaces generated by these curves can be obtained. In the geometric sense, some characterizations which have been obtained with the aid of mean and Gaussian curvatures of the rotational surfaces generated by these curves have been examined.

The benefits of this study for industrial design are:

1. With the help of computer, some new procedure for modelling some industrial objects can be obtained.

2. New ideas for producers about some object models industry in the form of geometric surface design so as to increase the choice of existing models previously can be provided.

Furthermore, by defining the cubic Hermitian and cubic Bezier curves in different spaces, such as Lorentz-Minkowski space, Galilean space and pseudo Galilean space, these curves and rotational surfaces can be investigated as a future work.

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