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Ruled Surfaces Constructed by Planar Curves in Euclidean 3-Space with Density

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Abstract

In the present study, firstly we recall the parametric expressions of planar curves with zero φ-curvature in Euclidean 3-space with density e^{ax_1} and with the aid of the Frenet frame of these planar curves, we obtain the Smarandache curves of them. After that, we study on ruled surfaces which are constructed by the curves with zero φ-curvature in Euclidean 3-space with density e^{ax_1} and their Smarandache curves by giving the striction curves, distribution parameters, mean curvature and Gaussian curvature of these ruled surfaces. Also, we give some examples for these surfaces by plotting their graphs. We use *Mathematica* when we are plotting the graphs of examples.

Keywords: Ruled surfaces, Smarandache curves, Weighted curvature.

1. Introduction

The curves and surfaces are popular topics studied in classical differential geometry and the problem of acquiring mean and Gaussian curvature of a hypersurface in the Euclidean and other spaces is one of the most important problems for geometers. Nowadays, manifold with density (or weighted manifold) is a new topic in geometry and it has been studied in many areas of mathematics, physics and economics. On the other hand, a ruled surface is a surface that can be swept out by moving a line in space and they can be used on different areas such as architectural, CAD, electric discharge machining and etc [1-3].

Furthermore, weighted manifold is a Riemannian manifold with positive density function e^{φ}. In 2003, Gromow [4] has introduced weighted curvature (or φ-curvature) κ_{φ} of a curve and weighted mean curvature (or φ-mean curvature) H_{φ} of an n-dimensional hypersurface on a manifold with density e^{φ}. Also, the generalizations of weighted curvature of a curve, weighted mean curvature and weighted Gaussian curvature (or φ-Gaussian curvature) G_{φ} of a Riemannian manifold with density e^{φ} has been given in [5]. After these definitions, lots of studies about the different characterizations of the curves and surfaces in different spaces with density have been done, for instance, [6-23] and etc.

In the present paper, striction curves, distribution parameters, mean and Gaussian curvatures of the ruled surfaces constructed by curves with zero weighted curvature in Euclidean 3-space with density and the Smarandache curves of them are obtained and some characterizations are given for them.

2. Preliminaries

Let $\alpha(u)$ be a planar curve given by $\alpha(u) = (x_1(u), x_2(u), 0)$. Then the Frenet frame $\{T, N, B\}$ and curvature κ of it in the Euclidean 3-space are [24].

$$
\begin{aligned}
T(u) &= \frac{1}{\sqrt{x_1'(u)^2 + x_2'(u)^2}}(x_1'(u), x_2'(u), 0),\\
N(u) &= \frac{1}{\sqrt{x_1'(u)^2 + x_2'(u)^2}}(-x_2'(u), x_1'(u), 0),\\
B(u) &= (0,0,1),\\
\kappa(u) &= \frac{x_1'(u)x_2''(u) - x_1''(u)x_2'(u)}{(x_1'(u)^2 + x_2'(u)^2)^{\frac{3}{2}}}.
\end{aligned}
\tag{2.1}
$$

Also, Smarandache curves which are introduced with the aid of Frenet frame of a curve is an important topic

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for differential geometry of curves and if we denote TN-Smarandache curve as γ_{TN}, TB-Smarandache curve as γ_{TB}, NB-Smarandache curve as γ_{NB} and TNB–Smarandache curve as γ_{TNB} of $\alpha(u)$, then they are defined as follows

$$\gamma_{TN}(u) = \frac{T(u)+N(u)}{\|T(u)+N(u)\|},\ \gamma_{TB}(u) = \frac{T(u)+B(u)}{\|T(u)+B(u))\|}, \qquad \textbf{(2.2)}$$

$$\gamma_{NB}(u) = \frac{N(u)+B(u)}{\|N(u)+N(u)\|} \text{ and } \gamma_{TNB}(u) = \frac{T(u)+N(u)+B(u)}{\|T(u)+N(u)+B(u)\|}.$$

The parametrization of

$$\varphi(u,v) = \alpha(u) + v.X(u),\ \ u,v \in I \subset \mathbb{R} \qquad \textbf{(2.3)}$$

is called a *ruled surface*, where the curve $\alpha(u)$ is *base curve* and $X(u)$ is *ruling* of it. The *striction curve* and *distribution parameter* of a ruled surface are given by

$$\beta(u) = \alpha(u) - \frac{\langle \alpha'(u), X'(u)\rangle}{\|X'(u)\|^2} X(u) \qquad \textbf{(2.4)}$$

and

$$\delta = \frac{det[\alpha'(u), X(u), X'(u)]}{\|X'(u)\|^2}, \qquad \textbf{(2.5)}$$

respectively [25,26]. Also, the distribution parameter gives a characterization for ruled surface and it is known that, the ruled surface whose distribution parameter vanishes is *developable*.

If κ and N are the curvature and the normal vector of a curve, respectively, then the *φ-curvature* κ_φ of the curve on a manifold with density e^φ is defined by [5]

$$\kappa_\varphi = \kappa - \frac{d\varphi}{dN}. \qquad \textbf{(2.6)}$$

3. Results and Discussion

3.1. Planar Curves with Zero φ-Curvature in E^3 with Density

In [27], authors have found that, the planar curves with zero φ-curvature in Euclidean space with density e^{ax_1}, $(a \neq 0)$ can be parameterized by

$$\alpha_1(u) = \left(x_1(u), c_2 \mp \frac{arctan\left(\sqrt{c_1 e^{2ax_1(u)} - 1}\right)}{a}, 0 \right)$$

or

$$\alpha_2(u) = \left(d_2 - \frac{ln(cos(d_1 + ax_2(u)))}{a}, x_2(u), 0 \right),$$

where, $c_1 > e^{-2ax_1(u)}$, $-\frac{\pi}{2} + 2k\pi < d_1 + ax_2(u) < \frac{\pi}{2} + 2k\pi$ and $c_1, c_2, d_1, d_2 \in \mathbb{R}$, $k \in \mathbb{Z}$.

So, the TN-Smarandache curve $\gamma_{1_{TN}}$, TB-Smarandache curve $\gamma_{1_{TB}}$, NB-Smarandache curve $\gamma_{1_{NB}}$ and TNB-Smarandache curve $\gamma_{1_{TNB}}$ of the curve $\alpha_1(u)$ are written as

$$\begin{aligned}
&\gamma_{1_{TN}}(u) \\
&= \left(\frac{\sqrt{-1 + c_1 e^{2ax_1(u)}} - 1}{\sqrt{2c_1 e^{2ax_1(u)}}}, \frac{\sqrt{-1 + c_1 e^{2ax_1(u)}} + 1}{\sqrt{2c_1 e^{2ax_1(u)}}}, 0 \right), \\
&\gamma_{1_{TB}}(u) = \left(\frac{\sqrt{-1 + c_1 e^{2ax_1(u)}}}{\sqrt{2c_1 e^{2ax_1(u)}}}, \frac{1}{\sqrt{2c_1 e^{2ax_1(u)}}}, \frac{1}{\sqrt{2}} \right), \qquad \textbf{(3.1)} \\
&\gamma_{1_{NB}}(u) = \left(\frac{-1}{\sqrt{2c_1 e^{2ax_1(u)}}}, \frac{\sqrt{-1 + c_1 e^{2ax_1(u)}}}{\sqrt{2c_1 e^{2ax_1(u)}}}, \frac{1}{\sqrt{2}} \right), \\
&\gamma_{1_{TNB}}(u) \\
&= \left(\frac{\sqrt{-1 + c_1 e^{2ax_1(u)}} - 1}{\sqrt{3c_1 e^{2ax_1(u)}}}, \frac{\sqrt{-1 + c_1 e^{2ax_1(u)}} + 1}{\sqrt{3c_1 e^{2ax_1(u)}}}, \frac{1}{\sqrt{3}} \right),
\end{aligned}$$

respectively and the TN-Smarandache curve $\gamma_{2_{TN}}$, TB-Smarandache curve $\gamma_{2_{TB}}$, NB-Smarandache curve $\gamma_{2_{NB}}$ and TNB-Smarandache curve $\gamma_{2_{TNB}}$ of the curve $\alpha_2(u)$ are written as

$$\begin{aligned}
\gamma_{2_{TN}}(u) &= \frac{1}{\sqrt{2}}(sin(d_1 + ax_2(u)) - cos(d_1 + ax_2(u)), \\
&\quad cos(d_1 + ax_2(u)) + sin(d_1 + ax_2(u)), 0), \\
\gamma_{2_{TB}}(u) &= \frac{1}{\sqrt{2}}(sin(d_1 + ax_2(u)), cos(d_1 + ax_2(u)), 1), \qquad \textbf{(3.2)} \\
\gamma_{2_{NB}}(u) &= \frac{1}{\sqrt{2}}(-cos(d_1 + ax_2(u)), sin(d_1 + \\
&\quad ax_2(u)), 1), \\
\gamma_{2_{TNB}}(u) &= \frac{1}{\sqrt{3}}(sin(d_1 + ax_2(u)) - cos(d_1 + ax_2(u)), \\
&\quad sin(d_1 + ax_2(u)) + cos(d_1 + ax_2(u)), 1),
\end{aligned}$$

respectively.

Furthermore, the results of the planar curves with zero φ-curvature in Euclidean space with density e^{bx_2} can be obtained with similar procedure to the planar curve with zero φ-curvature in Euclidean space with density e^{ax_1}.

3.2. Ruled Surfaces Constructed by Planar Curves in Euclidean 3-Space with Density

3.2.1. Ruled Surfaces Constructed by the curve $\alpha_1(u)$ and its Smarandache Curves

In this subsection, firstly we construct the ruled surfaces with the help of the curve $\alpha_1(u)$ and its Smarandache curves. Also, we obtain the mean curvatures, Gaussian curvatures, distribution parameters and striction curves for each of these ruled surfaces and give some characterizations for them.

Throughout this subsection, the base curves of ruled surfaces will be taken as the curve $\alpha_1(u)$.

If the ruling of the ruled surface is the TN-Smarandache curve $\gamma_{1_{TN}}(u)$ of the curve $\alpha_1(u)$, then from (2.3) and (3.1), the ruled surface $\varphi_{1_{TN}}(u,v)$ can be given by

$$\begin{aligned}\varphi_{1_{TN}}(u,v) &= \alpha_1(u) + v\gamma_{1_{TN}}(u)\\ &= \Bigg(x_1(u) + v\left(\frac{\sqrt{-1+c_1e^{2ax_1(u)}}-1}{\sqrt{2c_1e^{2ax_1(u)}}}\right),\\ &c_2 + \frac{\arctan\left(\sqrt{c_1e^{2ax_1(u)}-1}\right)}{a}\\ &+ v\left(\frac{\sqrt{-1+c_1e^{2ax_1(u)}}+1}{\sqrt{2c_1e^{2ax_1(u)}}}\right), 0\Bigg).\end{aligned}$$

Since the ruled surface $\varphi_{1_{TN}}$ is a parameterization of a plane, it is obvious that, the Gaussian and mean curvatures are zero and from (2.5), also the distribution parameter is zero and the surface is developable.

Also,
Theorem 3.2.1.1. *The base curve and the striction curve of $\varphi_{1_{TN}}$ never intersect.*

Proof. From (2.4), the striction curve $\beta_{1_{TN}}$ of $\varphi_{1_{TN}}$ is

$$\beta_{1_{TN}}(u) = \alpha_1(u) - \frac{\sqrt{c_1e^{2ax_1(u)}}}{\sqrt{2}a}\gamma_{1_{TN}}(u)$$

and this completes the proof.

Example. If we take $a=1$, $x_1(u)=\sin(u)$, $c_1=3$ and $c_2=5$ in the ruled surface $\varphi_{1_{TN}}$, we obtain that

$$\varphi_{1_{TN}}(u,v) = \Bigg(sin(u) + v\left(\frac{\sqrt{-1+3e^{2sin(u)}}-1}{\sqrt{6e^{2sin(u)}}}\right),$$

$$arctan\left(\sqrt{3e^{2sin(u)}-1}\right) + v\left(\frac{\sqrt{-1+3e^{2sin(u)}}+1}{\sqrt{6e^{2sin(u)}}}\right) + 5, 0\Bigg).$$

In the following figure, one see this ruled surface for $(u,v) \in \left(0,\frac{\pi}{2}\right) \times (-5,5)$.

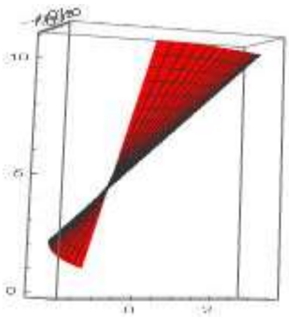

Figure 1. The ruled surface $\varphi_{1_{TN}}$.

If the ruling of the ruled surface is the TB-Smarandache curve $\gamma_{1_{TB}}(u)$ of the curve $\alpha_1(u)$, then from (2.3) and (3.1), the ruled surface $\varphi_{1_{TB}}(u,v)$ is parametrized by

$$\begin{aligned}\varphi_{1_{TB}}(u,v) &= \alpha_1(u) + v\gamma_{1_{TB}}(u)\\ &= \Bigg(x_1(u) + v\left(\frac{\sqrt{-1+c_1e^{2ax_1(u)}}}{\sqrt{2c_1e^{2ax_1(u)}}}\right),\\ &c_2 + \frac{arctan\left(\sqrt{c_1e^{2ax_1(u)}-1}\right)}{a}\\ &+ v\left(\frac{1}{\sqrt{2c_1e^{2ax_1(u)}}}\right), \frac{v}{\sqrt{2}}\Bigg).\end{aligned}$$

The Gaussian curvature and mean curvature of $\varphi_{1_{TB}}$ are

$$G = -\frac{a^2c_1e^{2ax_1}}{(c_1e^{2ax_1}+a^2v^2)^2}$$

and

$$H = \frac{a^2v\left(av\sqrt{c_1e^{2ax_1(u)}} - c_1\sqrt{2c_1e^{2ax_1(u)}-2}\right)}{\sqrt{2}\sqrt{c_1e^{2ax_1(u)}}(c_1e^{2ax_1(u)}+a^2v^2)^{3/2}},$$

respectively.

Also,
Theorem 3.2.1.2. ***i)*** *The ruled surface $\varphi_{1_{TB}}$ is not developable.*

ii) *The base curve and the striction curve of $\varphi_{1_{TB}}$ coincide.*

Proof. From (2.5), the distribution parameter of $\varphi_{1_{TB}}$ is

$$\delta_{1_{TB}} = \frac{\sqrt{c_1e^{2ax_1(u)}}}{a}.$$

Since $\delta_{1_{TB}}$ cannot be zero, $\varphi_{1_{TB}}$ is not developable. Also, from (2.4) the striction curve is

$$\beta_{1_{TB}}(u) = \alpha_1(u).$$

So, the proof completes.

Example. If we take $a=1$, $x_1(u)=\ln(u)$, $c_1=3$ and $c_2=5$ in the ruled surface $\varphi_{1_{TB}}$, we obtain that

$$\begin{aligned}\varphi_{1_{TB}}(u,v) &= \Bigg(ln(u) + v\left(\frac{\sqrt{-1+3e^{2ln(u)}}}{\sqrt{6e^{2ln(u)}}}\right),\\ &arctan\left(\sqrt{3e^{2ln(u)}-1}\right) + v\left(\frac{1}{\sqrt{6e^{2ln(u)}}}\right) + 5, \frac{v}{\sqrt{2}}\Bigg).\end{aligned}$$

The following figure shows the graphic of this ruled surface for $(u,v) \in \left(\frac{1}{\sqrt{3}}, 8\right) \times (-14,14)$.

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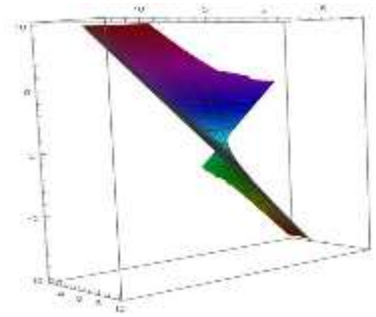

Figure 2. The ruled surface $\varphi_{1_{TB}}$.

Let the ruling curve of the ruled surface be the NB-Smarandache curve $\gamma_{1_{NB}}(u)$ of $\alpha_1(u)$. Thus from (2.3) and (3.1), the ruled surface $\varphi_{1_{NB}}(u,v)$ can be parametrized by

$$\begin{aligned}\varphi_{1_{NB}}(u,v) &= \alpha_1(u) + v\gamma_{1_{NB}}(u)\\ &= (x_1(u) + v\left(\frac{-1}{\sqrt{2c_1e^{2ax_1(u)}}}\right),\\ &\quad c_2 + \frac{arctan\left(\sqrt{c_1e^{2ax_1(u)}-1}\right)}{a}\\ &\quad + v\left(\frac{\sqrt{-1+c_1e^{2ax_1(u)}}}{\sqrt{2c_1e^{2ax_1(u)}}}\right), \frac{v}{\sqrt{2}}).\end{aligned}$$

The Gaussian curvature and mean curvature of $\varphi_{1_{NB}}$ are

$$G = 0$$

and

$$H = \frac{a\left(a^2\sqrt{c_1e^{2ax_1(u)}}v^2 + 2c_1e^{2ax_1(u)}\left(\sqrt{c_1e^{2ax_1(u)}} + \sqrt{2}av\right)\right)}{2\sqrt{c_1e^{2ax_1(u)}}\left(2c_1e^{2ax_1(u)} + av\left(2\sqrt{2}\sqrt{c_1e^{2ax_1(u)}} + av\right)\right)},$$

respectively.

Also,

Theorem 3.2.1.3. ***i)*** *The ruled surface $\varphi_{1_{NB}}$ is developable.*

ii) *The base curve and the striction curve of $\varphi_{1_{NB}}$ never intersect.*

Proof. From (2.5), the distribution parameter of $\varphi_{1_{NB}}$ is

$$\delta_{1_{NB}} = 0$$

and so, $\varphi_{1_{NB}}$ is developable. Also, from (2.4) the striction curve $\beta_{1_{NB}}(u)$ on $\varphi_{1_{NB}}$ is

$$\beta_{1_{NB}}(u) = \alpha_1(u) - \frac{\sqrt{2}\sqrt{c_1e^{2ax_1(u)}}}{a}\gamma_{1_{NB}}(u)$$

and this completes the proof.

Example. Taking $a = 1$, $x_1(u) = \frac{1}{u}$, $c_1 = 3$ and $c_2 = 5$ in the ruled surface $\varphi_{1_{NB}}$, we get

$$\begin{aligned}\varphi_{1_{NB}}(u,v) &= (1/u + v\left(\frac{-1}{\sqrt{6e^{2/u}}}\right),\\ &\quad arctan\left(\sqrt{3e^{2/u}-1}\right) + v\left(\frac{\sqrt{3e^{2/u}-1}}{\sqrt{6e^{2/u}}}\right) + 5, \frac{v}{\sqrt{2}}).\end{aligned}$$

Figure 3 shows the graphic of this ruled surface for $(u,v) \in (0.01,100) \times (-50,50)$.

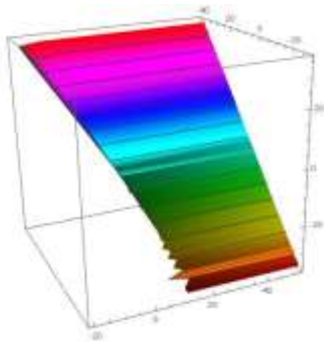

Figure 3. The ruled surface $\varphi_{1_{NB}}$.

Finally, let the ruling curve of the ruled surface be the TNB-Smarandache curve $\gamma_{1_{TNB}}(u)$ of the curve $\alpha_1(u)$. Thus from (2.3) and (3.1), the ruled surface $\varphi_{1_{TNB}}$ can be given by

$$\begin{aligned}\varphi_{1_{TNB}}(u,v) &= \alpha_1(u) + v\gamma_{1_{TNB}}(u)\\ &= (x_1(u) + v\left(\frac{\sqrt{-1+c_1e^{2ax_1(u)}}-1}{\sqrt{3c_1e^{2ax_1(u)}}}\right)\\ &\quad c_2 + \frac{arctan\left(\sqrt{c_1e^{2ax_1(u)}-1}\right)}{a}\\ &\quad + v\left(\frac{\sqrt{-1+c_1e^{2ax_1(u)}}+1}{\sqrt{3c_1e^{2ax_1(u)}}}\right), \frac{v}{\sqrt{3}}).\end{aligned}$$

The Gaussian curvature and mean curvature of $\varphi_{1_{TNB}}$ are

$$G = -\frac{a^2c_1e^{2ax_1(u)}}{4\left(c_1e^{2ax_1(u)} + av\left(\sqrt{3}\sqrt{c_1e^{2ax_1(u)}} + av\right)\right)^2}$$

and

$$H = \frac{a\left(\begin{array}{c}2a^2\sqrt{c_1e^{2ax_1(u)}}v^2+\\ c_1e^{2ax_1(u)}\left(\sqrt{c_1e^{2ax_1(u)}}-\sqrt{3}a\left(-2+\sqrt{-1+c_1e^{2ax_1(u)}}\right)v\right)\end{array}\right)}{4\sqrt{2}\sqrt{c_1e^{2ax_1(u)}}\left(c_1e^{2ax_1(u)}+av\left(\sqrt{3}\sqrt{c_1e^{2ax_1(u)}}+av\right)\right)^{3/2}},$$

respectively.

Also,

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Theorem 3.2.1.4. ***i)*** *The ruled surface $\varphi_{1_{TNB}}$ is not developable.*

ii) *The base curve and the striction curve of $\varphi_{1_{TNB}}$ never intersect.*

Proof. From (2.5), the distribution parameter of $\varphi_{1_{TNB}}$ is

$$\delta_{1_{TNB}} = \frac{\sqrt{c_1 e^{2ax_1(u)}}}{2a}$$

and from (2.4), the striction curve $\beta_{1_{TNB}}(u)$ on $\varphi_{1_{TNB}}$ is

$$\beta_{1_{TNB}}(u) = \alpha_1(u) - \frac{\sqrt{3}\sqrt{c_1 e^{2ax_1(u)}}}{2a}\gamma_{1_{TNB}}(u).$$

So, (i) and (ii) are obvious.

Example. If we take $a = 1$, $x_1(u) = \ln(\tan(u))$, $c_1 = 3$ and $c_2 = 5$ in this ruled surface, then we obtain

$$\begin{aligned}\varphi_{1_{TNB}}(u,v) = (&ln(tan(u)) \\ &+ v\left(\frac{\sqrt{-1+3e^{2\,ln(tan(u))}}-1}{\sqrt{9e^{2\,ln(tan(u))}}}\right), \\ &5 + arctan\left(\sqrt{3e^{2ln(tan(u))}-1}\right) \\ &+ v\left(\frac{\sqrt{-1+3e^{2ln(tan(u))}}+1}{\sqrt{9e^{2ln(tan(u))}}}\right), \frac{v}{\sqrt{3}}).\end{aligned}$$

Figure 4 shows this ruled surface for $(u,v) \in \left(\frac{7\pi}{6}, \frac{8\pi}{6}\right) \times (-1,1)$.

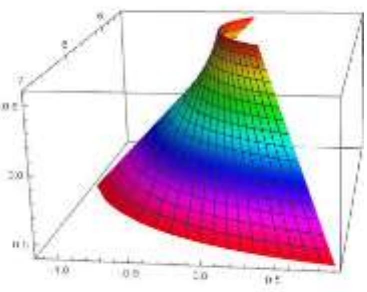

Figure 4. The ruled surface $\varphi_{1_{TNB}}$.

3.2.2. Ruled Surfaces Constructed by the curve $\alpha_2(u)$ and its Smarandache Curves

In this subsection, firstly we construct the ruled surfaces with the help of the curve $\alpha_2(u)$ and its Smarandache curves. Also, we obtain the mean curvatures, Gaussian curvatures, distribution parameters and striction curves for each of these ruled surfaces and give some characterizations for them.

Throughout this subsection, the base curves of ruled surfaces will be taken as the curve $\alpha_2(u)$.

If the ruling curve of the ruled surface is the TN-Smarandache curve $\gamma_{2_{TN}}(u)$ of the curve $\alpha_2(u)$, then from (2.3) and (3.2), the ruled surface $\varphi_{2_{TN}}(u,v)$ can be given by

$$\begin{aligned}\varphi_{2_{TN}}(u,v) &= \alpha_2(u) + v\gamma_{2_{TN}}(u) \\ &= (d_2 - \frac{\ln(\cos(d_1+ax_2(u)))}{a} + \\ &v\left(\frac{sin(d_1+ax_2(u)) - cos(d_1+ax_2(u))}{\sqrt{2}}\right), \\ &x_2(u) + v\left(\frac{cos(d_1+ax_2(u)) + sin(d_1+ax_2(u))}{\sqrt{2}}\right), 0).\end{aligned}$$

Since the ruled surface $\varphi_{2_{TN}}$ is a parametrization of a plane, it is obvious that, the Gaussian curvature and mean curvature are zero and from (2.5), the distribution parameter $\delta_{2_{TN}}$ of it is zero and so it is developable.

Also,
Theorem 3.2.2.1. *The base curve and the striction curve of $\varphi_{2_{TN}}$ never intersect.*

Proof. From (2.4), the striction curve $\beta_{2_{TN}}(u)$ on $\varphi_{2_{TN}}$ is

$$\beta_{2_{TN}}(u) = \alpha_2(u) - \frac{sec(d_1 + ax_2(u))}{\sqrt{2}a}\gamma_{2_{TN}}(u),$$

which completes the proof.

Example. Taking $a = -1$, $x_2(u) = u^2$, $d_1 = 3$ and $d_2 = 5$ in the ruled surface $\varphi_{2_{TN}}$, we get

$$\begin{aligned}&\varphi_{2_{TN}}(u,v) = \\ &(5 + ln(cos(3-u^2)) + v\left(\frac{sin(3-u^2) - cos(3-u^2)}{\sqrt{2}}\right), \\ &u^2 + v\left(\frac{cos(3-u^2) + sin(3-u^2)}{\sqrt{2}}\right), 0).\end{aligned}$$

Figure 5 shows the graphic of this ruled surface for $(u,v) \in \left(\sqrt{3-\frac{\pi}{2}}, \sqrt{3+\frac{\pi}{2}}\right) \times (-1,1)$.

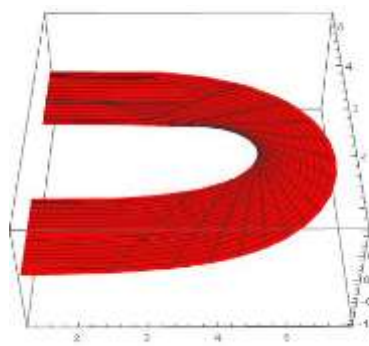

Figure 5. The ruled surface $\varphi_{2_{TN}}$

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If the ruling of the ruled surface is the TB-Smarandache curve $\gamma_{2_{TB}}(u)$ of $\alpha_2(u)$, then from (2.3) and (3.2), the ruled surface $\varphi_{2_{TB}}(u,v)$ can be given by

$$\begin{aligned}\varphi_{2_{TB}}(u,v) &= \alpha_2(u) + v\gamma_{2_{TB}}(u)\\ &= (d_2 - \tfrac{ln(cos(d_1+ax_2(u)))}{a} + v\left(\tfrac{sin(d_1+ax_2(u))}{\sqrt{2}}\right),\\ &\quad x_2(u) + v\left(\tfrac{cos(d_1+ax_2(u))}{\sqrt{2}}\right), \tfrac{v}{\sqrt{2}}).\end{aligned}$$

The Gaussian curvature and mean curvature of the ruled surface $\varphi_{2_{TB}}$ are

$$G = -\frac{4a^2\cos(d_1+ax_2(u))}{(2+a^2v^2+a^2v^2\cos(2(d_1+ax_2(u))))^2}$$

and

$$H = \frac{a^2v(av+av\cos(2(d_1+ax_2))-2\sqrt{2}\sin(d_1+ax_2))}{\sqrt{2}\sec(d_1+ax_2)(2+a^2v^2+a^2v^2\cos(2(d_1+ax_2)))^{3/2}},$$

respectively.

Also,
Theorem 3.2.2.2. ***i)*** *The ruled surface $\varphi_{2_{TB}}$ is not developable.*

ii) *The base curve and the striction curve of $\varphi_{2_{TB}}$ coincide.*

Proof. From (2.5), the distribution parameter of $\varphi_{2_{TB}}$ is

$$\delta_{2_{TB}} = \frac{sec(d_1+ax_2(u))}{2a}$$

and from (2.4), the striction curve $\beta_{2_{TB}}(u)$ on $\varphi_{2_{TB}}$ is

$$\beta_{2_{TB}}(u) = \alpha_2(u).$$

So, we have (i) and (ii).

Example. If we take $a=1$, $x_2(u)=u$, $d_1=3$ and $d_2=5$ in the ruled surface $\varphi_{2_{TB}}$, we obtain that

$$\begin{aligned}\varphi_{2_{TB}}(u,v) &= (5 - ln(cos(u+3)) + v\left(\tfrac{sin(u+3)}{\sqrt{2}}\right),\\ &\quad u + v\left(\tfrac{cos(u+3)}{\sqrt{2}}\right), \tfrac{v}{\sqrt{2}}).\end{aligned}$$

In figure 6, one see this ruled surface for $(u,v) \in \left(-3-\frac{\pi}{2}, -3+\frac{\pi}{2}\right) \times (-15,15)$.

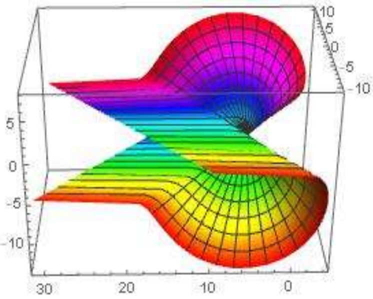

Figure 6. The ruled surface $\varphi_{2_{TB}}$

If the ruling of the ruled surface is the NB-Smarandache curve $\gamma_{2_{NB}}(u)$ of the curve $\alpha_2(u)$, then from (2.3) and (3.2), the ruled surface $\varphi_{2_{NB}}(u,v)$ can be given by

$$\begin{aligned}\varphi_{2_{NB}}(u,v) &= \alpha_2(u) + v\gamma_{2_{NB}}(u)\\ &=(d_2 - \tfrac{ln(cos(d_1+ax_2(u)))}{a} - v\left(\tfrac{cos(d_1+ax_2(u))}{\sqrt{2}}\right),\\ &\quad x_2(u) + v\left(\tfrac{\sin(d_1+ax_2(u))}{\sqrt{2}}\right), \tfrac{v}{\sqrt{2}}).\end{aligned}$$

The Gaussian curvature and mean curvature of the ruled surface $\varphi_{2_{NB}}$ are

$$G = 0$$

and

$$H=\frac{a(4+a^2v^2+4\sqrt{2}av\cos(d_1+ax_2(u))+a^2v^2\cos(2(d_1+ax_2(u))))}{4(2+2\sqrt{2}av\cos(d_1+ax_2(u))+a^2v^2\cos^2(2(d_1+ax_2(u))))^{3/2}},$$

respectively.

Also,
Theorem 3.2.2.3. ***i)*** *The ruled surface $\varphi_{2_{NB}}$ is developable.*

ii) *The base curve and the striction curve of $\varphi_{2_{NB}}$ never intersect.*

Proof. From (2.5), the distribution parameter of $\varphi_{2_{NB}}$ is

$$\delta_{2_{NB}} = 0.$$

So, $\varphi_{2_{NB}}$ is developable. Also, from (2.4) the striction curve $\beta_{2_{NB}}(u)$ on $\varphi_{2_{NB}}$ is

$$\beta_{2_{NB}}(u) = \alpha_2(u) - \frac{\sqrt{2}sec(d_1+ax_2(u))}{a}\gamma_{2_{NB}}(u)$$

and this completes the proof.

Example. Taking $a=1$, $x_2(u)=\ln(u)$, $d_1=3$ and $d_2=5$ in the ruled surface $\varphi_{2_{NB}}$, we get

$$\begin{aligned}\varphi_{2_{NB}}(u,v) &=(5 - ln(cos(ln(u)+3)) -\\ &\quad v\left(\tfrac{cos(ln(u)+3)}{\sqrt{2}}\right), ln(u) + v\left(\tfrac{sin(ln(u)+3)}{\sqrt{2}}\right), \tfrac{v}{\sqrt{2}}).\end{aligned}$$

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In the following figure, one see this ruled surface for $(u,v) \in \left(e^{-3-\frac{\pi}{2}}, e^{-3+\frac{\pi}{2}}\right) \times (-5,5)$.

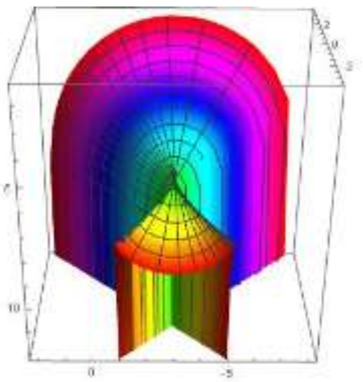

Figure 7. The ruled surface $\varphi_{2_{NB}}$

Finally, if the ruling of the ruled surface is the TNB-Smarandache curve $\gamma_{2_{TNB}}(u)$ of the curve $\alpha_2(u)$, then from (2.3) and (3.2), the ruled surface $\varphi_{2_{TNB}}(u,v)$ can be given by

$$\begin{aligned}\varphi_{2_{TNB}}(u,v) &= \alpha_2(u) + v\gamma_{2_{TNB}}(u)\\ &= (d_2 - \frac{\ln(\cos(d_1+ax_2(u)))}{a} + \\ &\quad v\left(\frac{sin(d_1+ax_2(u)) - cos(d_1+ax_2(u))}{\sqrt{3}}\right),\\ &\quad x_2(u) + v\left(\frac{cos(d_1+ax_2(u)) + sin(d_1+ax_2(u))}{\sqrt{3}}\right), \frac{v}{\sqrt{3}}).\end{aligned}$$

The Gaussian and mean curvatures of the ruled surface $\varphi_{2_{TNB}}$ are

$$G = -\frac{a^2 cos(d_1 + ax_2(u))}{\begin{pmatrix} 2 + a^2v^2 + 2\sqrt{3}avcos(d_1 + ax_2(u)) + \\ a^2v^2 cos(2(d_1 + ax_2(u))) \end{pmatrix}^2}$$

and

$$H = \frac{a\begin{pmatrix} 1 + 2\sqrt{3}avcos(d_1 + ax_2(u)) + \\ a^2v^2\, cos\left(2(d_1 + ax_2(u))\right) - \sqrt{3}avsin(d_1 + ax_2(u)) \end{pmatrix}}{2\begin{pmatrix} 2 + a^2v^2 + 2\sqrt{3}avcos(d_1 + ax_2(u)) + \\ a^2v^2\, cos\left(2(d_1 + ax_2(u))\right) \end{pmatrix}^{3/2}},$$

respectively.

Also,

Theorem 3.2.2.4. ***i)*** *The ruled surface $\varphi_{2_{TNB}}$ is not developable.*

ii) *The base curve and the striction curve of $\varphi_{2_{TNB}}$ never intersect.*

Proof. From (2.5), the distribution parameter of $\varphi_{2_{TNB}}$ is

$$\delta_{2_{TNB}} = \frac{\sqrt{3}sec(d_1 + ax_2(u))}{2a}$$

and from (2.4), the striction curve $\beta_{2_{TNB}}$ on the ruled surface $\varphi_{2_{TNB}}$ is

$$\beta_{2_{TNB}} = \alpha_2(u) - \frac{\sqrt{3}\sec(d_1+ax_2(u))}{2a}\gamma_{2_{TNB}}(u).$$

Thus, these equations prove (i) and (ii).

Example. If we take $a = 1$, $x_2(u) = u$, $d_1 = 3$ and $d_2 = 5$ in the ruled surface $\varphi_{2_{TNB}}$, we obtain that

$$\begin{aligned}&\varphi_{2_{TNB}}(u,v) =\\ &(5 - ln(cos(u+3)) + v\left(\frac{sin(u+3) - cos(u+3)}{\sqrt{3}}\right),\\ &\qquad u + v\left(\frac{cos(u+3) + sin(u+3)}{\sqrt{3}}\right), \frac{v}{\sqrt{3}}).\end{aligned}$$

In figure 8, one see this ruled surface for $(u,v) \in \left(-3-\frac{\pi}{2}, -3+\frac{\pi}{2}\right) \times (-1.5,1.5)$.

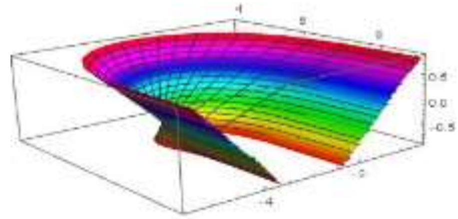

Figure 8. The ruled surface $\varphi_{2_{TNB}}$.

4. Conclusion

In the present study, we give some important results for ruled surfaces constructed by curves with zero φ-curvature in Euclidean 3-space with density. We hope that, this study will help to engineers and geometers who are dealing with surfaces in Euclidean space with density and in near future, this study can be tackled in Minkowski space, Galilean space and etc.

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Ethics

There are no ethical issues after the publication of this manuscript.

Authors' Contributions

All authors contributed equally to this manuscript and all authors reviewed the final manuscript.

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